

# UNIVERSAL TORSORS AND VALUES OF QUADRATIC POLYNOMIALS REPRESENTED BY NORMS

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ABSTRACT. Let  $K/k$  be an extension of number fields, and let  $P(t)$  be a quadratic polynomial over  $k$ . Let  $X$  be the affine variety defined by  $P(t) = N_{K/k}(\mathbf{z})$ . We study the Hasse principle and weak approximation for  $X$  in two cases. For  $[K : k] = 4$  and  $P(t)$  irreducible over  $k$  and split in  $K$ , we prove the Hasse principle and weak approximation. For  $k = \mathbb{Q}$  with arbitrary  $K$ , we show that the Brauer-Manin obstruction to the Hasse principle and weak approximation is the only one.

## CONTENTS

1. Introduction	1
2. Quadratic polynomials represented by a quartic norm	3
3. Universal torsors	4
4. Quadratic polynomials represented by a norm over $\mathbb{Q}$	9
References	11

## 1. INTRODUCTION

Let  $K/k$  be an extension of number fields of degree  $n$ . When can values of a polynomial  $P(t)$  over  $k$  be represented by norms of elements of  $K$ ? To answer this question, we study solutions  $(t, \mathbf{z}) \in k \times K$  of the equation

$$P(t) = N_{K/k}(\mathbf{z}). \quad (1)$$

This is closely related to the problem of studying the Hasse principle and weak approximation (see the end of this introduction for a review of this terminology) on a smooth proper model  $X^c$  of the affine hypersurface  $X \subset \mathbb{A}_k^1 \times \mathbb{A}_k^n$  with coordinates  $(t, \mathbf{z}) = (t, z_1, \dots, z_n)$  defined by (1), via a choice of a basis  $\omega_1, \dots, \omega_n$  of  $K$  over  $k$ , with  $N_{K/k}(\mathbf{z}) = N_{K/k}(z_1\omega_1 + \dots + z_n\omega_n)$ .

Colliot-Thélène conjectured that the Brauer–Manin obstruction to weak approximation is the only one on  $X^c$  (see [CT03]). This conjecture is known in the case where  $P(t)$  is constant, thanks to work of Sansuc [San81]; if additionally  $K/k$  is cyclic, it is known that the Hasse principle (proved by Hasse himself [Has30, p. 150]) and weak approximation hold. Other known cases of Colliot-Thélène’s conjecture, in some cases leading to a proof of the Hasse principle and weak approximation, include the class of Châtelet surfaces ( $[K : k] = 2$  and  $\deg(P(t)) \leq 4$ ) [CTSsanSD87a], [CTSsanSD87b], a class of singular cubic hypersurfaces ( $[K : k] = 3$  and  $\deg(P(t)) \leq 3$ )

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[CTSa189] and the case where  $K/k$  is arbitrary and  $P(t)$  is split over  $k$  with at most two distinct roots [HBSko02], [CTHarSko03], [SJ11]. Finally, if one admits Schinzel's hypothesis, then the conjecture is known for  $K/k$  cyclic and  $P(t)$  arbitrary [CTSkoSD98]. See [CT03, Introduction] and [BHB11, Section 1] for a more detailed discussion of these results and the difficulties of this problem.

In a recent preprint, Browning and Heath-Brown proved the conjecture for  $k = \mathbb{Q}$ ,  $[K : \mathbb{Q}] = 4$  and  $\deg(P(t)) = 2$  with  $P(t)$  irreducible over  $k$  and split over  $K$ . Their main result [BHB11, Theorem 1] answers a question of Colliot-Thélène (see [CTHarSko03, Section 2]) in the case  $k = \mathbb{Q}$ . We give a very short and elementary proof of this result for an arbitrary number field  $k$ . It is independent of the work of Browning and Heath-Brown and uses the fibration method in a simple and classical case.

**Theorem 1.** *Let  $P(t)$  be a quadratic polynomial that is irreducible over a number field  $k$  and split in  $K$  with  $[K : k] = 4$ . Then the Hasse principle and weak approximation hold for the variety  $X \subset \mathbb{A}_k^5$  defined by (1).*

If the ground field is  $\mathbb{Q}$ , we can prove a much more general result based on the analytic work of Browning and Heath-Brown in [BHB11, Theorem 2] and the descent method of Colliot-Thélène and Sansuc:

**Theorem 2.** *Let  $k = \mathbb{Q}$  and  $K$  be any number field. Let  $P(t) \in \mathbb{Q}[t]$  be an arbitrary quadratic polynomial. Then the Brauer–Manin obstruction to the Hasse principle and weak approximation is the only obstruction on any smooth proper model of  $X \subset \mathbb{A}_{\mathbb{Q}}^{n+1}$  defined by (1).*

Let  $X$  be the variety defined by (1) and let  $U \subset X$  be the open subvariety given by  $P(t) \neq 0$ . We will prove that the variety  $Y$  defined by [BHB11, (1.5)] is the restriction  $\mathcal{T}_U$  of a universal torsor  $\mathcal{T}$  over  $X$  to  $U$ , or a product of  $\mathcal{T}_U$  with a quasi-split torus. For the variety  $Y$ , [BHB11, Theorem 2] proves weak approximation using sieve methods from analytic number theory, inspired by work of Fouvry and Iwaniec [FI97]. While one step in Browning's and Heath-Brown's deduction of [BHB11, Theorem 1] from [BHB11, Theorem 2] leads to their restriction to  $[K : \mathbb{Q}] = 4$ , the combination of their analytic work with descent theory gives our more general Theorem 2. We also generalize Theorem 2 to a large class of multivariate polynomials  $P(t_1, \dots, t_\ell) \in \mathbb{Q}[t_1, \dots, t_\ell]$ .

**Terminology.** For an algebraic variety  $Z$  defined over a number field  $k$ , one says that the Hasse principle holds if  $\prod_{v \in \Omega_k} Z(k_v) \neq \emptyset$  (where  $\Omega_k$  is the set of places of  $k$  and  $k_v$  is the completion of  $k$  at  $v$ ) implies  $Z(k) \neq \emptyset$ . One says that weak approximation holds if  $Z(k)$  is dense in  $\prod_{v \in \Omega_k} Z(k_v)$  with the product topology, via the diagonal embedding.

If  $Z$  is smooth and proper, one says that the Brauer–Manin obstruction to the Hasse principle is the only one if  $(\prod_{v \in \Omega_k} Z(k_v))^{\text{Br}(Z)} \neq \emptyset$  implies that  $Z(k) \neq \emptyset$ , and that the Brauer–Manin obstruction to weak approximation is the only one if  $Z(k)$  is dense in  $(\prod_{v \in \Omega_k} Z(k_v))^{\text{Br}(Z)}$ . Here  $(\prod_{v \in \Omega_k} Z(k_v))^{\text{Br}(Z)}$  is the set of all  $(z_v) \in \prod_{v \in \Omega_k} Z(k_v)$  satisfying  $\sum_{v \in \Omega_k} \text{inv}_v(A(z_v)) = 0$  for each  $A$  in the Brauer group  $\text{Br}(Z) = H_{\text{ét}}^2(Z, \mathbb{G}_m)$  of  $Z$ , where the map  $\text{inv}_v : \text{Br}(k_v) \rightarrow \mathbb{Q}/\mathbb{Z}$  is the invariant map from local class field theory.

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## 2. QUADRATIC POLYNOMIALS REPRESENTED BY A QUARTIC NORM

In this section, we give a very short proof of Theorem 1 that is independent of the work of Browning and Heath-Brown [BHB11] and generalizes it from  $\mathbb{Q}$  to an arbitrary number field  $k$ . Let  $K/k$  be an extension of degree 4. Let  $P(t) \in k[t]$  be an irreducible quadratic polynomial that is split in  $K$ . Using a change of variables if necessary, we can assume that  $P(t) = c(t^2 - a)$ , with  $c \in k^\times$ , where  $a \in k^\times$  is not a square and  $\sqrt{a} \in K$ . Write  $L = k(\sqrt{a}) \subset K$ .

*Proof of Theorem 1.* Let  $U = \{(t, \mathbf{z}) : P(t) \neq 0\} \subset X$ . Let  $S \subset \mathbb{A}_k^2$  be the conic defined by the affine equation  $c = N_{L/k}(\mathbf{w})$  and let  $p : U \rightarrow S$  be the morphism defined by

$$(t, \mathbf{z}) \mapsto (t - \sqrt{a})^{-1} N_{K/L}(\mathbf{z}).$$

The morphism  $p$  is smooth.

Let  $S^c$  be a smooth compactification of  $S$ . Then there exists a smooth compactification  $X^c$  of  $X$  such that  $p$  extends to  $X^c \rightarrow S^c$ . The conic  $S$  satisfies weak approximation. We can assume that  $X^c$  has points everywhere locally; otherwise there is nothing to prove. This implies that  $S^c(k) \neq \emptyset$  and even  $S(k) \neq \emptyset$ . The fiber of  $p$  over a rational point  $\mathbf{w} \in S(k)$  is defined by the equation  $t - \sqrt{a} = \mathbf{w}^{-1} N_{K/L}(\mathbf{x})$ . This  $k$ -variety is isomorphic to a smooth quadric of dimension 3: writing  $\mathbf{x} = (x_1 + x_2\sqrt{a}) + (x_3 + x_4\sqrt{a})\beta$  for some  $\beta \in L$  such that  $K = L(\beta)$ , Lemma 1 below shows that

$$\mathbf{w}^{-1} N_{K/L}(\mathbf{x}) = f_0(x_1, \dots, x_4) + f_1(x_1, \dots, x_4)\sqrt{a}$$

for some quadratic forms  $f_0$  and  $f_1$  (over  $k$ ) of rank 4, which depend on  $\mathbf{w}$ . Hence the fiber is isomorphic to the affine  $k$ -hypersurface given by an equation  $f_1(x_1, \dots, x_4) = -1$ , which is indeed a smooth quadric of dimension 3. Therefore, Theorem 1 holds by [CTSanSD87a, Proposition 3.9].  $\square$

Now we show that the quadratic forms in the proof of Theorem 1 have rank 4.

**Lemma 1.** *Let  $K/k$  be a quartic extension of fields of characteristic 0 which contains a quadratic subextension  $L = k(\sqrt{a})$ . Take  $\beta \in K$  with  $K = L(\beta)$ . Let  $\rho \in L^\times$ . Let  $\mathbf{y} = (y_1 + y_2\sqrt{a}) + (y_3 + y_4\sqrt{a})\beta$  be a  $K$ -variable. If*

$$\rho N_{K/L}(\mathbf{y}) = f_0(y_1, \dots, y_4) + f_1(y_1, \dots, y_4)\sqrt{a},$$

*then the quadratic forms  $f_0$  and  $f_1$  have rank 4.*

*Proof.* Write  $\beta = \sqrt{u + v\sqrt{a}}$  for some  $u, v \in k$ . Elementary computations give

$$N_{K/L}(\mathbf{y}) = g_0(y_1, \dots, y_4) + g_1(y_1, \dots, y_4)\sqrt{a}$$

with

$$\begin{aligned} g_0(y_1, \dots, y_4) &= y_1^2 + ay_2^2 - u(y_3^2 + ay_4^2 - 2avy_3y_4), \\ g_1(y_1, \dots, y_4) &= 2y_1y_2 - 2uy_3y_4 - v(y_3^2 + ay_4^2). \end{aligned}$$

Multiplying by  $\rho = \rho_0 + \rho_1\sqrt{a} \neq 0$  (where  $\rho_0, \rho_1 \in k$ ), we see that  $f_0$  and  $f_1$  are of the form  $\lambda g_0 + \mu g_1$  for some  $(\lambda, \mu) \in k^2$  with  $(\lambda, \mu) \neq (0, 0)$ . Then

$$\lambda g_0(y_1, \dots, y_4) + \mu g_1(y_1, \dots, y_4) = q_0(y_1, y_2) + q_1(y_3, y_4)$$

with

$$\begin{aligned} q_0(y_1, y_2) &= \lambda y_1^2 + 2\mu y_1y_2 + a\lambda y_2^2, \\ q_1(y_3, y_4) &= -(\lambda u + \mu v)y_3^2 - 2(\lambda av + \mu u)y_3y_4 - a(\lambda u + \mu v)y_4^2. \end{aligned}$$

Clearly  $q_0$  and  $q_1$  have rank 2 since  $a \notin k^{\times 2}$  implies that

$$\text{disc}(q_0) = \lambda^2 a - \mu^2 \neq 0, \quad \text{disc}(q_1) = -(\lambda^2 a - \mu^2)(v^2 a - u^2) \neq 0.$$

The result follows.  $\square$

**Remark 1.** The analog of Theorem 1 holds for global fields of positive characteristic different from 2 as well. Indeed, it is not hard to see that our arguments and the proof of [CTSanSD87a, Proposition 3.9] remain valid for such fields.

### 3. UNIVERSAL TORSORS

The basic strategy is based on the following result, which reduces the problem of the Hasse principle and weak approximation on a variety to the same questions on its universal torsors, where we have no Brauer–Manin obstructions. This kind of result has been proved essentially by Colliot-Thélène and Sansuc in their seminal paper [CTSan87]. However, they developed their theory under the simplifying assumption that the varieties involved are proper. Skorobogatov developed a variant under less stringent assumptions in [Sko99]. Descent on open varieties also features in [CTSko00] and [CT03]. We will use the following variant:

**Proposition 1.** *Let  $Z$  be a smooth, geometrically rational variety over a number field  $k$  with algebraic closure  $\bar{k}$ . Let  $\bar{Z} = Z \times_k \bar{k}$ . Assume furthermore that  $\bar{k}[Z]^\times = \bar{k}^\times$ , that  $\text{Pic}(\bar{Z})$  is free of finite rank, that  $Z$  has universal torsors and that there is an open subvariety  $U \subset Z$  such that the restriction  $\mathcal{T}_U$  satisfies weak approximation for any universal  $Z$ -torsor  $\mathcal{T}$ . Then the Brauer–Manin obstruction to the Hasse principle and weak approximation is the only one for any smooth proper model  $Z^c$  of  $Z$ .*

The condition  $\bar{k}[Z]^\times = \bar{k}^\times$  means that the only invertible regular functions on  $Z$  are the constant ones.

Let us explain how to obtain this result from the existing results in the literature. Let  $(x_v) \in (\prod_v Z^c(k_v))^{\text{Br}(Z^c)}$ . For any finite set  $S$  of places of  $k$ , we must find an  $x \in Z^c(k)$  that is arbitrarily close to  $x_v$  for all  $v \in S$ .

Because of our assumptions on  $Z$ ,  $\text{Br}_1(Z)/\text{Br}_0(Z) \cong H^1(k, \text{Pic}(\bar{Z}))$  is finite, where  $\text{Br}_1(Z)$  is the kernel of the natural map  $\text{Br}(Z) \rightarrow \text{Br}(\bar{Z})$  and  $\text{Br}_0(Z)$  is the image of  $\text{Br}(k) \rightarrow \text{Br}(Z)$ . We note that the variety  $Z^c$  is smooth, proper and geometrically rational, so that  $\text{Br}(\bar{Z}^c) = 0$ . Therefore,

we can apply [CTSk00, Proposition 1.1] to conclude that  $Z(\mathbb{A}_k)^{\text{Br}_1(Z)}$  is dense in  $(\prod_v Z^c(k_v))^{\text{Br}(Z^c)}$ , and we can choose  $(y_v) \in Z(\mathbb{A}_k)^{\text{Br}_1(Z)}$  such that  $y_v$  is as close as we wish to  $x_v$  for all  $v \in S$ .

By assumption, we can find a universal torsor  $f : \mathcal{T} \rightarrow Z$  and an adelic point  $(t_v) \in \mathcal{T}(\mathbb{A}_k) \subset \prod_v \mathcal{T}(k_v)$  such that  $f((t_v)) = (y_v)$  using descent theory [Sko99, Theorem 3]. Since  $\mathcal{T}$  is smooth over  $X$ , the implicit function theorem implies that there exists  $(u_v) \in \prod_v \mathcal{T}_U(k_v)$  such that  $u_v$  is arbitrarily close to  $t_v$  for all places  $v \in S$ .

As weak approximation holds on  $\mathcal{T}_U$  by assumption, we find  $u \in \mathcal{T}_U(k)$  such that  $u$  is as close as we wish to  $u_v$  for all  $v \in S$ . Since  $f$  is continuous, the point  $x = f(u)$  has the required properties.

The main result of this section is concerned with the existence of universal torsors [CTSan87, (2.0.4)] over  $X$  as in (1) and their local description.

Let us recall some more definitions. If  $k$  is a field and if  $A$  is an étale  $k$ -algebra, then the  $k$ -variety  $R_{A/k}(\mathbb{G}_{\text{m},A})$  is defined via its functor of points: take  $R_{A/k}(\mathbb{G}_{\text{m},A})(B) = (A \otimes_k B)^\times$  functorially for every  $k$ -algebra  $B$ . The norm map  $N_{A/k}$  is defined as in [Bou58, §12.2]. We denote the absolute Galois group of  $k$  by  $\Gamma_k$ .

**Proposition 2.** *Let  $K/k$  be an extension of fields of degree  $n$ . Let  $P(t)$  be an irreducible separable polynomial of degree  $r$  over  $k$ .*

*The variety  $X \subset \mathbb{A}_k^{n+1}$  defined by (1) is smooth and geometrically integral, with  $\text{Pic}(\overline{X})$  free of finite rank and  $\overline{k}[X]^\times = \overline{k}^\times$ . Let  $U$  be the open subset of  $X$  defined by  $P(t) \neq 0$ . Then  $\text{Pic}(\overline{U}) = 0$ .*

*Let  $c \in k^\times$  be the leading coefficient of  $P(t)$ , let  $L$  be the field  $k[t]/(P(t))$  and let  $\eta$  be the class of  $t$  in  $L$ . Let  $A = L \otimes_k K$ . For any universal torsor  $\mathcal{T}$  over  $X$ , there exists a solution  $(\rho, \xi) \in L^\times \times K^\times$  of the equation  $cN_{L/k}(\rho) = N_{K/k}(\xi)$  such that  $\mathcal{T}_U$  (its restriction to  $U$ ) is isomorphic to the subvariety of  $\mathbb{A}_k^1 \times R_{A/k}(\mathbb{G}_{\text{m},A})$  (with coordinates  $(t, \mathbf{z})$ ) given by the equation*

$$t - \eta = \rho N_{A/L}(\mathbf{z}). \quad (2)$$

Using only the basic definitions, it is easy to see that one can specialize equation (2) as follows in the two “extreme” cases:

- (a) If  $P(t)$  splits completely in  $K$ , then  $\mathcal{T}_U$  is isomorphic to the subvariety of  $\mathbb{A}_k^1 \times (R_{K/k}(\mathbb{G}_{\text{m},K}))^r$  (with coordinates  $(t, \mathbf{x}_1, \dots, \mathbf{x}_r)$ ) given by the equation

$$t - \eta = \rho \prod_{i=1}^r \sigma_i^{-1}(N_{K/\sigma_i(L)}(\mathbf{x}_i)) \quad (3)$$

where  $\sigma_1, \dots, \sigma_r$  is a set of representatives of  $\Gamma_k/\Gamma_L$ .

- (b) If  $P(t)$  remains irreducible in  $K$ , then  $\mathcal{T}_U$  is isomorphic to the subvariety of  $\mathbb{A}_k^1 \times R_{F/k}(\mathbb{G}_{\text{m},F})$  (with coordinates  $(t, \mathbf{x})$ ) given by the equation

$$t - \eta = \rho N_{F/L}(\mathbf{x}) \quad (4)$$

where  $F = L \cdot K$ .

The proof of Proposition 2 will occupy most of the remainder of this section. The  $\bar{k}$ -variety  $\bar{X}$  can be described by an equation of the form

$$c \prod_{i=1}^r (t - \eta_i) = u_1 \cdots u_n \quad (5)$$

where  $\eta_1, \dots, \eta_r$  are the embeddings of  $\eta$  in  $\bar{k}$ . We note that  $X$  is smooth because  $P(t)$  is separable. Consider the morphism  $p : X \rightarrow \mathbb{A}_k^1$  given by  $(t, \mathbf{x}) \mapsto t$ . Over  $\bar{k}$ , it has precisely  $r$  reducible fibers  $X_i$ , for  $i = 1, \dots, r$ , over  $t = \eta_i$ . Each of these has  $n$  irreducible components  $D_{i,j} = \{t = \eta_i, u_j = 0\}$  for  $j = 1, \dots, n$ . Let  $U_0$  be the open subset of  $\mathbb{A}_k^1$  where  $P(t) \neq 0$  and let  $U = p^{-1}(U_0) \subset X$ . We have

$$\bar{U} = U \times_k \bar{k} \cong (\mathbb{A}_k^1 \setminus \{\eta_1, \dots, \eta_r\}) \times \mathbb{G}_{m, \bar{k}}^{n-1},$$

so that  $\text{Pic}(\bar{U}) = 0$ .

We have  $\bar{k}[X]^\times = \bar{k}^\times$ . Indeed, the generic fiber of  $\bar{X} \rightarrow \mathbb{A}_k^1$  is  $\mathbb{G}_{m, \bar{k}(t)}^{n-1}$ . Therefore, any  $f \in \bar{k}[X]^\times$  has the form  $f = g(t)u_1^{m_1} \cdots u_n^{m_n}$  with  $g \in k(t)$  and  $m_1, \dots, m_n \in \mathbb{Z}$ . If  $g(t)$  has a root or pole in some  $t_0 \notin \{\eta_1, \dots, \eta_r\}$ , then  $f$  or  $f^{-1}$  is not regular in a point on  $p^{-1}(t_0)$ . Otherwise, we have

$$g(t) = c' \prod_{i=1}^r (t - \eta_i)^{e_i}$$

for some  $c' \in \bar{k}^\times$  and  $e_1, \dots, e_r \in \mathbb{Z}$ . Then

$$\text{div}(f) = \sum_{i=1}^r \sum_{j=1}^n (e_i + m_j) D_{i,j},$$

so  $f \in \bar{k}[X]^\times$  if and only if  $e_1 = \dots = e_r = -m_1 = \dots = -m_n$ . By (5), this is equivalent to saying that  $f$  is a constant in  $\bar{k}^\times$ .

By descent theory [CTSan87, Corollary 2.3.4], universal torsors over  $X$  exist if and only if the exact sequence of  $\Gamma_k$ -modules

$$1 \rightarrow \bar{k}^\times \rightarrow \bar{k}[U]^\times \rightarrow \bar{k}[U]^\times / \bar{k}^\times \rightarrow 1 \quad (6)$$

is split.

It is easy to see that the abelian group  $\bar{k}[U]^\times / \bar{k}^\times$  is free of rank  $r + n - 1$ , generated by the classes of the functions  $t - \eta_1, \dots, t - \eta_r, u_1, \dots, u_n$  with an obvious  $\Gamma_k$ -action and the relation

$$\sum_{i=1}^r [t - \eta_i] - \sum_{j=1}^n [u_j] = 0 \quad (7)$$

because of the equation defining  $X$ .

The exact sequence (6) is split if and only if the classes can be lifted to  $\bar{k}[U]^\times$  in a  $\Gamma_k$ -equivariant way, via a map

$$\phi : \bar{k}[U]^\times / \bar{k}^\times \rightarrow \bar{k}[U]^\times, \quad [t - \eta] \mapsto \rho^{-1}(t - \eta), \quad [u_1] \mapsto \xi^{-1}u_1 \quad (8)$$

where  $\rho \in L^\times$  and  $\xi \in K^\times$ . Because of the unique relation (7), the pair  $(\rho, \xi) \in L^\times \times K^\times$  defines such a splitting if and only if

$$cN_{L/k}(\rho) = N_{K/k}(\xi). \quad (9)$$

We now want to apply [CTSan87, Theorem 2.3.1, Corollary 2.3.4] for the local description of universal torsors over  $X$ . We will describe a morphism of tori  $d : M \rightarrow T$  such that its dual map of characters fits into the following commutative diagram of  $\Gamma_k$ -equivariant homomorphisms.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widehat{T} & \xrightarrow{\widehat{d}} & \widehat{M} & \longrightarrow & \text{Pic}(\overline{X}) \longrightarrow 0 \\
& & \sim \downarrow i & & \sim \downarrow j & & \parallel \\
1 & \longrightarrow & \overline{k}[U]^\times / \overline{k}^\times & \xrightarrow{\text{div}} & \text{Div}_{\overline{X} \setminus \overline{U}}(\overline{X}) & \longrightarrow & \text{Pic}(\overline{X}) \longrightarrow 0
\end{array} \tag{10}$$

Here, the second row is exact because  $\text{Pic}(\overline{U}) = 0$  and  $\overline{k}[X]^\times = \overline{k}^\times$ .

The  $\Gamma_k$ -module  $\overline{k}[U]^\times / \overline{k}^\times$  is isomorphic to the module of characters of the algebraic  $k$ -torus  $T \subset R_{L/k}(\mathbb{G}_{m,L}) \times R_{K/k}(\mathbb{G}_{m,K})$  with coordinates  $(\mathbf{z}_1, \mathbf{z}_2)$  given by

$$N_{L/k}(\mathbf{z}_1) = N_{K/k}(\mathbf{z}_2). \tag{11}$$

Indeed, the character group  $\widehat{T}$  is the quotient of  $\mathbb{Z}[\Gamma_k/\Gamma_L] \oplus \mathbb{Z}[\Gamma_k/\Gamma_K]$  with the diagonal  $\Gamma_k$ -action by the relation

$$\sum_{\sigma\Gamma_L \in \Gamma_k/\Gamma_L} \sigma\Gamma_L = \sum_{\gamma\Gamma_K \in \Gamma_k/\Gamma_K} \gamma\Gamma_K.$$

The isomorphism  $i : \widehat{T} \rightarrow \overline{k}[U]^\times / \overline{k}^\times$  is given by

$$i(\sigma\Gamma_L) = [t - \sigma(\eta)], \quad i(\gamma\Gamma_K) = [\gamma(u_1)].$$

The abelian group  $\text{Div}_{\overline{X} \setminus \overline{U}}(\overline{X})$  is free of rank  $rn$ , generated by  $D_{i,j}$  for  $i = 1, \dots, r$  and  $j = 1, \dots, n$ . There is a bijection  $\Gamma_k/\Gamma_L \times \Gamma_k/\Gamma_K \rightarrow \{D_{i,j}\}$  defined by  $(\sigma\Gamma_L, \gamma\Gamma_K) \mapsto \{t = \sigma(\eta), \gamma(u_1) = 0\}$  that is compatible with the action of  $\Gamma_k$ , acting diagonally on the left hand side. Recalling  $A = L \otimes_k K$ , this shows that  $\text{Div}_{\overline{X} \setminus \overline{U}}(\overline{X})$  is isomorphic to the module of characters of the  $k$ -torus  $M = R_{A/k}(\mathbb{G}_{m,A})$ . Let  $j : \widehat{M} \rightarrow \text{Div}_{\overline{X} \setminus \overline{U}}(\overline{X})$  be this isomorphism.

Consider the homomorphism  $\text{div} : \overline{k}[U]^\times / \overline{k}^\times \rightarrow \text{Div}_{\overline{X} \setminus \overline{U}}(\overline{X})$  that maps a function to its divisor. We have

$$\text{div}([t - \eta]) = \sum_{j=1}^n D_{1,j}, \quad \text{div}([u_1]) = \sum_{i=1}^r D_{i,1}.$$

Now  $\text{div}$  induces a homomorphism on the character modules  $\widehat{d} : \widehat{T} \rightarrow \widehat{M}$ . The dual of this homomorphism is then given by the morphism of  $k$ -tori

$$d : M \rightarrow T, \quad \mathbf{z} \mapsto (N_{A/L}(\mathbf{z}), N_{A/K}(\mathbf{z})).$$

Let  $S$  be the Néron–Severi torus dual to the  $\Gamma_k$ -module  $\text{Pic}(\overline{X})$ , so that we have an exact sequence of tori

$$1 \rightarrow S \rightarrow M \rightarrow T \rightarrow 1.$$

This makes  $M$  into a  $T$ -torsor under  $S$ .

We now describe the map  $U \rightarrow T$  induced by the splitting  $\phi$  as in (8) by a choice of  $(\rho, \xi) \in L^\times \times K^\times$  satisfying (9). The induced map is given by

$$U \rightarrow T, \quad (t, \mathbf{x}) \mapsto (\rho^{-1}(t - \eta), \xi^{-1}\mathbf{x}),$$

and it is easy to see that the image is in  $T$  using the equation of  $X$  and the condition (9). Therefore, the image of  $U$  in  $T$  is isomorphic to the subvariety of  $\mathbb{A}_k^1 \times T$  with coordinates  $(t, \mathbf{z}_1, \mathbf{z}_2)$  defined by

$$t - \eta = \rho \mathbf{z}_1.$$

By [CTSan87, Theorem 2.3.1, Corollary 2.3.4], any universal torsor  $\mathcal{T}_U$  over  $U$  is the pullback of a torsor  $M$  from  $T$  to  $U$ . Our computations show that it is isomorphic to the subvariety of  $\mathbb{A}_k^1 \times R_{A/k}(\mathbb{G}_{m,A})$  with coordinates  $(t, \mathbf{z})$  defined by (3). This completes the proof of Proposition 2.  $\square$

**Remark 2.** One can determine equations for universal torsors  $\mathcal{T}$  over the smooth locus  $X_{\text{sm}}$  of the variety  $X$  defined by (1) even if  $P(t)$  is not irreducible over  $k$ ; note that  $X$  is not smooth if  $P(t)$  is not separable. Then  $\text{Pic}(X_{\text{sm}})$  is a finitely generated (but not necessarily free) abelian group. So  $\mathcal{T}$  will be a torsor over  $X_{\text{sm}}$  under the group of multiplicative type that is dual to  $\text{Pic}(X_{\text{sm}})$ .

The result is as follows: Assume that

$$P(t) = cP_1(t)^{e_1} \cdots P_d(t)^{e_d}$$

for  $c \in k^\times$ , some irreducible monic polynomials  $P_i(t) \in k[t]$  and positive integers  $e_i$ . Write  $L_i = k[t]/(P_i(t))$  and let  $\eta_i$  be the class of  $t$  in  $L_i$ . For  $i = 1, \dots, d$ , consider the étale  $L_i$ -algebra  $A_i = L_i \otimes_k K$ . Let  $U \subset X_{\text{sm}}$  be the open subvariety given by  $P(t) \neq 0$ . For any universal torsor  $\mathcal{T}$  over  $X_{\text{sm}}$ , there exists a solution  $(\rho_1, \dots, \rho_d, \xi) \in L_1^\times \times \cdots \times L_d^\times \times K^\times$  of the equation

$$cN_{L_1/k}(\rho_1)^{e_1} \cdots N_{L_d/k}(\rho_d)^{e_d} = N_{K/k}(\xi)$$

such that  $\mathcal{T}_U$  is isomorphic to the subvariety of  $\mathbb{A}_k^1 \times \prod_{i=1}^d R_{A_i/k}(\mathbb{G}_{m,A_i})$  with coordinates  $(t, \mathbf{z}_1, \dots, \mathbf{z}_d)$  given by the system of equations

$$t - \eta_i = \rho_i N_{A_i/L_i}(\mathbf{z}_i) \quad \text{for } 1 \leq i \leq d.$$

The proof is a straightforward generalization of the proof of Proposition 2.

Note that [HBSko02, Theorem 2.2] is a special case of this result.

In case that  $k$  is a number field, the following result links the existence of universal torsors as in Proposition 2 to the absence of Brauer–Manin obstructions on  $X$ .

This can also be deduced from general results ([Sko01, Proposition 6.1.4] and [CTSko00, Proposition 1.1]). However, our more elementary proof easily generalizes to the setting of Remark 2, where these general results do not apply.

**Lemma 2.** *Let  $P(t)$  be an irreducible polynomial over a number field  $k$ . Let  $K/k$  be an extension of finite degree  $n$ . Let  $X \subset \mathbb{A}_k^{n+1}$  be the variety defined by (1). A universal torsor over  $X$  exists if there is no Brauer–Manin obstruction to the Hasse principle on a smooth proper model of  $X$ .*

*Proof.* Consider the variety  $E \subset R_{L/k}(\mathbb{G}_{m,L}) \times R_{K/k}(\mathbb{G}_{m,K})$  defined by the equation  $cN_{L/k}(\mathbf{z}_1) = N_{K/k}(\mathbf{z}_2)$ , corresponding to the equality (9). We have a natural map  $U \rightarrow E$  defined by  $(t, \mathbf{x}) \mapsto (t - \eta, \mathbf{x})$ . It is clear that  $E$  is a principal homogeneous space of the torus  $T \subset R_{L/k}(\mathbb{G}_{m,L}) \times R_{K/k}(\mathbb{G}_{m,K})$  defined by  $N_{L/k}(\mathbf{z}_1) = N_{K/k}(\mathbf{z}_2)$ .



By Hironaka's theorem, there exist a smooth compactification  $E^c$  of  $E$ , a smooth compactification  $U^c$  of  $U$  and a morphism  $U^c \rightarrow E^c$  extending the map  $U \rightarrow E$  above. Our assumption implies

$$\left( \prod_v U^c(k_v) \right)^{\text{Br}(U^c)} \neq \emptyset \quad \text{and hence} \quad \left( \prod_v E^c(k_v) \right)^{\text{Br}(E^c)} \neq \emptyset.$$

Since the Brauer–Manin obstruction to weak approximation is the only one for compactifications of homogeneous spaces of linear algebraic groups [San81, Theorem 8.12], we have  $E(k) \neq \emptyset$ , i.e., there exists  $(\rho, \xi) \in L^\times \times K^\times$  satisfying (9). Now the existence of a universal torsor over  $X$  follows from [CTSan87, Corollary 2.3.4].  $\square$

#### 4. QUADRATIC POLYNOMIALS REPRESENTED BY A NORM OVER $\mathbb{Q}$

Let  $k = \mathbb{Q}$ . As before, we can assume without loss of generality that  $P(t) = c(t^2 - a)$  with  $c \in \mathbb{Q}^\times$  and  $a \in \mathbb{Q}$ , but now we do not assume that  $P(t)$  is split in  $K$ . Using the deep work of Browning and Heath-Brown and our description of universal torsors, we can prove the following result:

**Proposition 3.** *If the quadratic polynomial  $P(t)$  is irreducible over  $\mathbb{Q}$ , then the restriction  $\mathcal{T}_U$  of each universal torsor  $\mathcal{T}$  over  $X$  as in Proposition 2 satisfies weak approximation.*

*Proof.* Assume that  $P(t)$  is split in  $K$ . Consider  $\mathcal{T}_U \subset \mathbb{A}_k^1 \times (R_{K/k}(\mathbb{G}_{m,K}))^2$  defined by equation (3) in the case  $r = 2$ . For any  $\sigma \in \Gamma_k$ , we have  $\sigma(L) = L$ , and for any  $x \in L$ , we have  $\sigma(x) = \sigma^{-1}(x)$ . Therefore, (3) can be rewritten as

$$t - \sqrt{a} = \rho N_{K/L}(\mathbf{x}_1) \cdot \sigma(N_{K/L}(\mathbf{x}_2)), \quad (12)$$

where  $\sigma \in \Gamma_k$  with  $\sigma(\sqrt{a}) = -\sqrt{a}$ .

The variety determined by this equation is isomorphic to the subvariety  $Y$  of  $\mathbb{A}_k^1 \times (R_{K/k}(\mathbb{G}_{m,K}))^2$  defined by the equation

$$N_{K/k}(\mathbf{w})(t - \sqrt{a}) = \rho N_{K/L}(\mathbf{y}), \quad (13)$$

via the substitution

$$\mathbf{w} = \mathbf{x}_2^{-1}, \quad \mathbf{y} = \mathbf{x}_1 \mathbf{x}_2^{-1}$$

with inverse

$$\mathbf{x}_1 = \mathbf{w}^{-1} \mathbf{y}, \quad \mathbf{x}_2 = \mathbf{w}^{-1}$$

using  $N_{K/k}(\mathbf{x}_2) = N_{L/k}(N_{K/L}(\mathbf{x}_2)) = N_{K/L}(\mathbf{x}_2) \cdot \sigma(N_{K/L}(\mathbf{x}_2))$ . This is exactly [BHB11, equation (1.5)]. Weak approximation then holds on  $Y$  because of [BHB11, Theorem 2].

Assume now that  $P(t)$  remains irreducible over  $K$  and write  $F = K \cdot L$ , where  $L = k(\sqrt{a})$ . Choose some  $\sigma \in \Gamma_K$  such that  $\sigma \notin \Gamma_F = \Gamma_L \cap \Gamma_K$ , so  $\sigma \notin \Gamma_L$ . Therefore,  $\sigma$  is a representative of the non-trivial class both in  $\Gamma_K/\Gamma_F$  and in  $\Gamma_k/\Gamma_L$ .

Let  $\gamma_1, \dots, \gamma_n$  be a set of coset representatives of  $\Gamma_L/\Gamma_F$ . We claim that a set of representatives of  $\Gamma_k/\Gamma_F$  is given by  $\gamma_1, \dots, \gamma_n, \gamma_1\sigma, \dots, \gamma_n\sigma$ . Indeed, if  $\gamma_i\sigma\Gamma_F = \gamma_j\sigma\Gamma_F$ , then we have  $\sigma^{-1}\gamma_j^{-1}\gamma_i\sigma \in \Gamma_F = \Gamma_L \cap \Gamma_K$ . Since  $L/k$  is Galois, this gives  $\gamma_j^{-1}\gamma_i \in \sigma\Gamma_L\sigma^{-1} = \Gamma_{\sigma(L)} = \Gamma_L$ , so  $\gamma_j^{-1}\gamma_i \in \sigma\Gamma_K\sigma^{-1} = \Gamma_K$  since  $\sigma \in \Gamma_K$ . Hence  $\gamma_j^{-1}\gamma_i \in \Gamma_L \cap \Gamma_K = \Gamma_F$ , so  $\gamma_i\Gamma_F = \gamma_j\Gamma_F$ , which implies

$i = j$ . Furthermore, if  $\gamma_i \sigma \Gamma_F = \gamma_j \Gamma_F$ , then  $\gamma_j^{-1} \gamma_i \sigma \in \Gamma_F \subset \Gamma_L$ , which contradicts the fact that  $\gamma_i, \gamma_j \in \Gamma_L$ , but  $\sigma \notin \Gamma_L$ . Finally,  $\gamma_i \Gamma_F = \gamma_j \Gamma_F$  only for  $i = j$  by construction. This proves the claim.

Therefore,  $N_{F/k}(\mathbf{w}) = N_{F/L}(\mathbf{w}) N_{F/L}(\sigma(\mathbf{w}))$ . We note that  $\sigma$  induces an automorphism over  $k$  of the variety  $R_{F/k}(\mathbb{G}_{m,F})$ : this is clear from the functor-of-points description of  $R_{F/k}(\mathbb{G}_{m,F})$ .

Using this observation, we see that the variety  $Y' \subset \mathbb{A}_k^1 \times (R_{F/k}(\mathbb{G}_{m,F}))^2$  with coordinates  $(t, \mathbf{w}, \mathbf{y})$  defined by

$$N_{F/k}(\mathbf{w})(t - \sqrt{a}) = \rho N_{F/L}(\mathbf{y})$$

(i.e. equation (13) with  $K$  replaced by  $F$ ) is isomorphic to the product  $\mathcal{T}_U \times R_{F/k}(\mathbb{G}_{m,F})$  with coordinates  $(t, \mathbf{x}, \mathbf{y})$  subject to (4). The isomorphism is defined by the map

$$(t, \mathbf{w}, \mathbf{y}) \mapsto (t, (\mathbf{w} \sigma(\mathbf{w}))^{-1} \mathbf{y}, \mathbf{w}),$$

the inverse substitution being given by

$$(t, \mathbf{x}, \mathbf{y}) \mapsto (t, \mathbf{y}, \mathbf{x} \mathbf{y} \sigma(\mathbf{y})).$$

Since  $Y'$  satisfies weak approximation by [BHB11, Theorem 2] and since  $R_{F/k}(\mathbb{G}_{m,F})$  is rational and therefore has non-trivial  $k_v$ -points for any place  $v$ , this implies that  $\mathcal{T}_U$  satisfies weak approximation.  $\square$

*Proof of Theorem 2.* If  $P(t)$  is split over  $\mathbb{Q}$  with two distinct roots, then Theorem 2 is a special case of [HBSko02, Theorem 1.1]. If it is split over  $\mathbb{Q}$  with one double root,  $U \subset X$  as in Proposition 2 is a principal homogeneous space of a torus, and Theorem 2 holds by [San81].

Next, assume that  $P(t)$  is irreducible over  $\mathbb{Q}$ . Assume that there is no Brauer–Manin obstruction to the Hasse principle on a smooth and proper model of  $X$ . Then Lemma 2 shows that universal torsors  $\mathcal{T}$  over  $X$  exist. By Proposition 3,  $\mathcal{T}_U$  satisfies weak approximation. Proposition 2 shows that  $\bar{k}[X]^\times = \bar{k}^\times$  and that  $\text{Pic}(\bar{X})$  is free of finite rank. Then an application of Proposition 1 gives the result.  $\square$

**Corollary 1.** *If the quadratic polynomial  $P(t) \in \mathbb{Q}[t]$  is not split in the Galois closure of  $K/\mathbb{Q}$ , then the Hasse principle and weak approximation hold on any smooth proper model of  $X \subset \mathbb{A}_{\mathbb{Q}}^{n+1}$  defined by (1).*

*Proof.* By [Wei12, Theorem 2.2], the smooth proper model  $X^c$  satisfies  $\text{Br}(X^c) = \text{Br}_0(X^c)$ , so the result follows immediately from Theorem 2.  $\square$

Finally, we generalize Theorem 2 to equations involving a multivariate polynomial  $P(t_1, \dots, t_\ell)$ , using techniques developed by Harari in [Har97]:

**Corollary 2.** *Let  $P_0, P_1, P_2$  be polynomials in  $\ell - 1$  variables  $t_2, \dots, t_\ell$  over  $\mathbb{Q}$  of arbitrary degree satisfying*

$$\gcd(P_0(t_2, \dots, t_\ell), P_1(t_2, \dots, t_\ell), P_2(t_2, \dots, t_\ell)) = 1.$$

*Let  $K$  be an arbitrary number field of degree  $n = [K : \mathbb{Q}]$ . Then the Brauer–Manin obstruction to the Hasse principle and weak approximation is the only obstruction on any smooth proper model of  $X \subset \mathbb{A}_{\mathbb{Q}}^{n+\ell}$  defined by the equation*

$$t_1^2 \cdot P_2(t_2, \dots, t_\ell) + t_1 \cdot P_1(t_2, \dots, t_\ell) + P_0(t_2, \dots, t_\ell) = N_{K/\mathbb{Q}}(\mathbf{x}).$$

*Proof.* Consider the projection  $\pi : X \rightarrow \mathbb{A}_{\mathbb{Q}}^{\ell-1}$  defined by  $(\mathbf{t}, \mathbf{x}) \mapsto (t_2, \dots, t_{\ell})$  and consider the closed subset

$$F = \{P_0(t_2, \dots, t_{\ell}) = P_1(t_2, \dots, t_{\ell}) = P_2(t_2, \dots, t_{\ell}) = 0\}$$

of  $\mathbb{A}_{\mathbb{Q}}^{\ell-1}$ , which is of codimension at least 2 by assumption.

The fibers of  $\pi$  over  $\mathbb{A}_{\mathbb{Q}}^{\ell-1} \setminus F$  are geometrically integral. The fiber over each rational point in this set is defined by  $P(t_1) = N_{K/\mathbb{Q}}(\mathbf{z})$  for some non-zero polynomial  $P(t_1)$  of degree at most 2. By Theorem 2 for quadratic  $P(t_1)$ , by rationality for linear  $P(t_1)$  and by [San81] for constant  $P(t_1)$ , this has the property that the Brauer–Manin obstruction to the Hasse principle and weak approximation is the only obstruction on any smooth proper model. The generic fiber of  $\pi$  is a rational variety. Therefore, the result follows by an application of [Har97, Théorème 3.2.1].  $\square$

## REFERENCES

- [BHB11] T. D. Browning and D. R. Heath-Brown. Quadratic polynomials represented by norm forms. *Geom. Funct. Anal.*, to appear, arXiv:1109.0232, 2011.
- [Bou58] N. Bourbaki. *Éléments de mathématique. 23. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre 8: Modules et anneaux semi-simples*. Actualités Sci. Ind. no. 1261. Hermann, Paris, 1958.
- [CT03] J.-L. Colliot-Thélène. Points rationnels sur les fibrations. In *Higher dimensional varieties and rational points (Budapest, 2001)*, volume 12 of *Bolyai Soc. Math. Stud.*, pages 171–221. Springer, Berlin, 2003.
- [CTHarSko03] J.-L. Colliot-Thélène, D. Harari, and A. N. Skorobogatov. Valeurs d'un polynôme à une variable représentées par une norme. In *Number theory and algebraic geometry*, volume 303 of *London Math. Soc. Lecture Note Ser.*, pages 69–89. Cambridge Univ. Press, Cambridge, 2003.
- [CTSal89] J.-L. Colliot-Thélène and P. Salberger. Arithmetic on some singular cubic hypersurfaces. *Proc. London Math. Soc.* (3), 58(3):519–549, 1989.
- [CTSan87] J.-L. Colliot-Thélène and J.-J. Sansuc. La descente sur les variétés rationnelles. II. *Duke Math. J.*, 54(2):375–492, 1987.
- [CTSanSD87a] J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer. Intersections of two quadrics and Châtelet surfaces. I. *J. reine angew. Math.*, 373:37–107, 1987.
- [CTSanSD87b] J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer. Intersections of two quadrics and Châtelet surfaces. II. *J. reine angew. Math.*, 374:72–168, 1987.
- [CTSko00] J.-L. Colliot-Thélène and A. N. Skorobogatov. Descent on fibrations over  $\mathbf{P}_k^1$  revisited. *Math. Proc. Cambridge Philos. Soc.*, 128(3):383–393, 2000.
- [CTSkoSD98] J.-L. Colliot-Thélène, A. N. Skorobogatov, and P. Swinnerton-Dyer. Rational points and zero-cycles on fibred varieties: Schinzel's hypothesis and Salberger's device. *J. reine angew. Math.*, 495:1–28, 1998.
- [FI97] E. Fouvry and H. Iwaniec. Gaussian primes. *Acta Arith.*, 79(3):249–287, 1997.
- [Har97] D. Harari. Flèches de spécialisations en cohomologie étale et applications arithmétiques. *Bull. Soc. Math. France*, 125(2):143–166, 1997.
- [Has30] H. Hasse. Die Normenresttheorie relativ-Abelscher Zahlkörper als Klassenkörpertheorie im Kleinen. *J. reine angew. Math.*, 162:145–154, 1930.
- [HBSko02] D. R. Heath-Brown and A. Skorobogatov. Rational solutions of certain equations involving norms. *Acta Math.*, 189(2):161–177, 2002.
- [San81] J.-J. Sansuc. Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres. *J. reine angew. Math.*, 327:12–80, 1981.

- [SJ11] M. Swarbrick Jones. A Note On a Theorem of Heath-Brown and Skorobogatov, arXiv:1111.4089, 2011.
- [Sko99] A. N. Skorobogatov. Beyond the Manin obstruction. *Invent. Math.*, 135(2):399–424, 1999.
- [Sko01] A. N. Skorobogatov. *Torsors and rational points*, volume 144 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.
- [Wei12] D. Wei. On the equation  $N_{K/k}(\Xi) = P(t)$ , preprint, 2012.

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